

THE CONORMAL DERIVATIVE PROBLEM FOR EQUATIONS OF VARIATIONAL TYPE IN NONSMOOTH DOMAINS

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ABSTRACT. It is well known that elliptic boundary value problems in smooth domains have smooth solutions, but if the domain is, say, C^1 , the solutions need not be Lipschitz. Recently Korevaar has identified a class of Lipschitz domains, in which solutions of the capillary problem are Lipschitz assuming the contact angle relates correctly to the geometry of the domain. Lipschitz bounds for more general boundary value problems in the same class of domains are proved. Applications to variational inequalities are also considered.

According to the standard theory [5, 10], solutions of elliptic boundary problems in $C^{k,\alpha}$ domains ($k \geq 2$) are also $C^{k,\alpha}$ up to the boundary of the domain; more recent work [2, 4, 11, 16, 17, 19] has shown that this statement is true for $k = 1$ also. However, if the domain is C^1 , even if all other data of the problem are C^∞ , the solution need not even be Lipschitz (see [8]). As can be seen for harmonic functions in a sector, for Lipschitz domains the connection between smoothness of the solution and that of the domain is more delicate. Here we investigate this connection for a family of boundary value problems modelled on the capillary problem in a domain with corners. We prove gradient bounds for this family of problems via a combination of Korevaar's approach to the capillary problem in nonsmooth domains [9, §4] and the author's gradient estimates in smooth domains [12, 21] which, in turn, are based on the interior estimates of Simon [28] and the global estimates of Ural'tseva for the capillary problem [29]. The important point is to verify that the considerations of those works apply to our situation.

Specifically, for Ω a bounded domain in \mathbb{R}^n with unit inner normal γ and appropriate functions $A^1, \dots, A^n, B : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$, and $\varphi : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$, we consider the problem

$$(0.1) \quad \begin{aligned} \operatorname{div} A(x, u, Du) + B(x, u, Du) &= 0 \quad \text{in } \Omega, \\ A(x, u, Du) \cdot \gamma + \varphi(x, u) &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

Our primary hypotheses are: Ω is Lipschitz and satisfies a uniform exterior sphere condition, $\partial\Omega$ is C^2 except for a small set, and $A(x, z, p) = \partial F(x, z, p)/\partial p$ for a scalar function F which is convex in p . The conditions on Ω come from [9] while the existence of F is the definition of variational

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type. The latter hypothesis was used in [12] and then dispensed with in [21] for smooth domains. Its present relevance will be discussed later. Of course the convexity of F guarantees that the matrix $(\frac{\partial A^i}{\partial p_j})$ is positive definite and hence the boundary value problem is elliptic.

Various technical hypotheses will be made on the structure of the functions F, A, B, φ , but we note here one particular such hypothesis for the capillary problem in two dimensions. In this case $F = (1 + |p|^2)^{1/2}$ and $A = (1 + |p|^2)^{-1/2}p$, so $|A| < 1$. Therefore a necessary condition for (0.1) to have smooth solutions is that $|\varphi| < 1$. If also φ is Lipschitz with $\varphi_z \leq 0$, it follows from [21, Exampie 1, p. 57] that $|Du|$ is bounded for domains with smooth boundaries. (When $\varphi_z = 0$, this result is due to Gerhardt [3] and Ural'tseva [29].) At a convex corner, even for constant φ , the condition $|\varphi| < 1$ is not strong enough to give a bound on $|Du|$ or, in fact, $|u|$. (Details of this situation are given in, e.g., [1, §5.2; 9, p. 20].) One crucial condition which gives these bounds is the existence of smooth functions $\bar{\varphi}$ (scalar) and $\bar{\gamma}$ (vector) with $|\bar{\varphi}| < 1$, $|\bar{\gamma}| \leq 1$, and $\varphi = \bar{\gamma} \cdot \gamma \bar{\varphi}$. Korevaar introduced this condition in [9], and he called $\bar{\gamma}$ a *pseudonormal*, a term we also use here.

An interesting comparison can be made between the results here and those in other works of the author. Suppose Ω is a cube in \mathbb{R}^3 , $\varphi = 0$, $A^i(x, z, p) = a^{ij}p_j$ for some constant positive-definite matrix (a^{ij}) , and B is $C^\infty(\bar{\Omega})$ and depends only on x . The results of [20] or [22] give a constant $\alpha \in (0, 1)$ determined only by (a^{ij}) such that solutions of (0.1) are $C^{1,\alpha}$ uniformly on any compact subset of $\bar{\Omega}$ minus its vertices. Global Hölder continuity of the solution is classical, and results in this paper will show global Lipschitz continuity if (a^{ij}) is symmetric.

Unfortunately some symmetry is crucial to the present approach. As in all gradient estimates, the key step is to find a function of the gradient which is a subsolution of a suitable boundary value problem. (For example if u is harmonic, then $|Du|^2$ is subharmonic; see [27] for further discussion in case a boundary gradient estimate is known.) In our case we consider a function $v_0(x, u, Du)$ which is convex and of quadratic growth in Du . (For technical reasons, it will be convenient to use linear rather than quadratic growth in the main body of this paper, but the quadratic growth is useful for illustration.) In addition, v_0 must have a suitable form to take advantage of the structure of the differential equation and the boundary condition. Among other things we would like the vector field $\frac{\partial v_0}{\partial p}$ to be tangential to the boundary. Then, writing a^{ij} for $\frac{\partial A^i}{\partial p_j}$ and differentiating the boundary condition in the tangential direction $\frac{\partial v_0}{\partial p}$ yields

$$a^{ij}\gamma_i D_{jk}u \frac{\partial v_0}{\partial p_k} = \text{lower order terms}$$

or

$$a^{ij}\gamma_i D_j v_0 = \text{lower order terms}$$

on $\partial\Omega$ and then the differential equation gives

$$D_i(a^{ij}D_j v_0) = \text{lower order terms}$$

in Ω . Therefore v_0 satisfies a conormal derivative problem, which leads to estimates on v_0 (and hence on $|Du|$) after appropriate estimation of the lower order terms.

To return to the role of symmetry, we suppose now for simplicity that $A^i(x, z, p) = a^{ij}p_j$ for some constant positive-definite matrix a^{ij} , with $a^{21} = 0$, that $\varphi \equiv 0$, and that Ω is two dimensional. Also, suppose first also that $\partial\Omega$ is the hyperplane $\{x^2 = 0\}$. As noted in [15], we can take

$$v_0 = p_1^2 + p_2^2$$

since $\frac{\partial v_0}{\partial p_2} = 0$ on $\partial\Omega$. Suppose next that $\partial\Omega$ is the angle $\{x^2 = 0, x^1 \geq 0\} \cup \{x^1 = 0, x^2 > 0\}$. If $v_0 = b^{ij}p_i p_j$ for some constant positive-definite (b^{ij}) , which we may assume symmetric, the boundary condition on $\{x^2 = 0\}$ and $\frac{\partial v_0}{\partial p}$ being tangential give

$$p_2 = 0 \quad \text{implies} \quad b^{21}p_1 + b^{22}p_2 = 0$$

and hence $b^{21} = 0$ since p_1 is arbitrary. Thus on $\{x^1 = 0\}$ we have

$$a^{11}p_1 + a^{12}p_2 = 0 \quad \text{implies} \quad b^{11}p_1 = 0,$$

which is only possible if $a^{12} = 0$, i.e., (a^{ij}) is symmetric. Such a hypothesis is not needed in the case of a domain with smooth boundary. Thus we see that the present approach is not easily modified to remove the symmetry hypothesis on a^{ij} (or, equivalently, the variational hypothesis on A).

We begin in §1 with some basic inequalities which are used to prove estimates on solutions of (0.1) in §2 and on their gradients in §3. Analogous results for parabolic problems are stated in §4. Examples in §5 illustrate our structure conditions and the role of the pseudonormal, and connections with variational inequalities appear in §6. Finally the progression from our estimates to regularity results is made, via existence theorems, in §7.

In general our notation follows [5, 12]. Unless otherwise indicated the arguments of A , B , and their derivatives are taken to be (x, u, Du) and hence omitted. Similarly $\varphi = \varphi(x, u)$. The subscripts x, z, p denote derivatives with respect to the appropriate scalar or vector variables. Other subscripts denote derivatives with respect to components of x or components of a vector or tensor, and superscripts denote derivatives with respect to components of p or components of a vector or tensor. The meaning should be clear from the context, e.g., $A_k^i = \frac{\partial A^i}{\partial x^k}$, $B^j = \frac{\partial B}{\partial p_j}$.

1. PRELIMINARIES

We begin by proving two inequalities which relate certain boundary and interior integrals. These inequalities are analogs of the standard ones proved as Lemma 2.1 in [12], but additional complications arise when investigating nonsmooth domains. A complete understanding of these complications is intimately tied to several of our structure conditions, so the results in this section will not be as independent of the gradient estimates as those in [12, §2].

Throughout we assume Ω to be a Lipschitz domain and $\bar{\gamma}$ to be a $C^{0,1}(\bar{\Omega})$ vector field with $|\bar{\gamma}| \leq 1$ in Ω and $\bar{\gamma} \cdot \gamma \geq 0$ almost everywhere with respect to surface measure ds on $\partial\Omega$.

Lemma 1.1. *If $K = \sup_{\Omega}(\operatorname{div} \bar{\gamma})_-$, then, for all nonnegative $h \in W^{1,1}$,*

$$(1.1) \quad \int_{\partial\Omega} h \gamma \cdot \bar{\gamma} ds \leq \int_{\Omega} |Dh| dx + K \int_{\Omega} h dx.$$

Proof.

$$\int_{\partial\Omega} h\gamma \cdot \bar{\gamma} ds = - \int_{\Omega} \operatorname{div}(h\bar{\gamma}) dx = - \int_{\Omega} Dh \cdot \bar{\gamma} dx - \int_{\Omega} h \operatorname{div} \bar{\gamma} dx. \quad \square$$

Note that such a $\bar{\gamma}$ can be constructed for an arbitrary $C^{0,1}$ domain very simply. Locally $\partial\Omega$ is the graph of a Lipschitz function, so we can choose $\bar{\gamma}$ locally to be constant (in the obvious direction) and then patch these together via a smooth partition of unity.

Our next lemma involves the boundary condition from (0.1) and some additional notation. For $u \in W^{2,1} \cap C^{0,1}(\bar{\Omega})$ and $h \in W^{1,1}$ we define

$$(1.2) \quad v = \sqrt{1 + |Du|^2}, \quad \nu = Du/v, \quad g^{ij} = \delta^{ij} = \delta^{ij} - \nu_i \nu_j, \\ \delta_i h = g^{ij} D_j h, \quad d\mu = v dx,$$

and for a $C^{0,1}$ vector function A defined on $\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^n$, we define

$$(1.3) \quad a^{ij} = v \frac{\partial A^i}{\partial p_j}(x, u, Du).$$

Lemma 1.2. *Suppose*

$$(1.4) \quad A(x, u, Du) \cdot \gamma + \varphi(x, u) = 0 \quad \text{on } \partial\Omega$$

and there is a function $\bar{\varphi}$ such that

$$(1.5) \quad |\bar{\varphi}| |Du| \leq \beta_1 Du \cdot A(x, u, Du) \quad \text{and} \quad \varphi = \bar{\varphi} \gamma \cdot \bar{\gamma} \quad \text{on } \partial\Omega$$

for some constant $\beta_1 \in [0, 1)$. If K is a constant such that

$$(1.6) \quad K \geq |D\bar{\gamma}| \quad \text{in } \Omega,$$

then, for any nonnegative $h \in W^{1,1}$, we have

$$(1.7) \quad \int_{\partial\Omega} h\bar{\gamma} \cdot \gamma Du \cdot \frac{A}{|A|} ds \\ \leq C(\beta_1) \left[\int_{\Omega} (|\delta h| + Kh) d\mu \right. \\ \left. + \int_{\Omega} h \left(\nu \cdot \bar{\gamma} D_i \left(\frac{A^i}{|A|} \right) - \bar{\gamma}^i \nu_j D_i \left(\frac{A^j}{|A|} \right) \right) d\mu \right].$$

Proof. Set

$$I = \int_{\partial\Omega} h \left[Du \cdot \frac{A}{|A|} \bar{\gamma} - Du \cdot \bar{\gamma} \frac{A}{|A|} \right] \cdot \gamma ds.$$

(This integral, in disguised form, was used in the proof of [30, Lemma 2].) The inequality in (1.5) gives

$$(1.8) \quad I \geq (1 - \beta_1) \int_{\partial\Omega} h\bar{\gamma} \cdot \gamma Du \cdot \frac{A}{|A|} ds.$$

The divergence theorem implies that

$$\begin{aligned}
I &= - \int_{\Omega} \left(Du \cdot \frac{A}{|A|} \bar{\gamma} - Du \cdot \bar{\gamma} \frac{A}{|A|} \right) \cdot Dh \, dx \\
&\quad - \int_{\Omega} h \left(\frac{A^j}{|A|} D_{ij} u \bar{\gamma}^i - \bar{\gamma}^j D_{ij} u \frac{A^i}{|A|} \right) \, dx \\
&\quad - \int_{\Omega} h \left(Du \cdot \frac{A}{|A|} \operatorname{div} \bar{\gamma} - \frac{A^i}{|A|} D_i \bar{\gamma}^j D_j u \right) \, dx \\
&\quad - \int_{\Omega} h \left(\bar{\gamma}^i D_j u D_i \left(\frac{A^j}{|A|} \right) - \bar{\gamma} \cdot Du D_i \left(\frac{A^i}{|A|} \right) \right) \, dx \\
&= - \int_{\Omega} \left(\nu \cdot \frac{A}{|A|} \bar{\gamma} - \nu \cdot \bar{\gamma} \frac{A}{|A|} \right) \cdot \delta h \, d\mu \\
&\quad - \int_{\Omega} h \left(\nu \cdot \frac{A}{|A|} \operatorname{div} \bar{\gamma} - \frac{A^i}{|A|} D_i \bar{\gamma}^j \nu_j \right) \, d\mu \\
&\quad + \int_{\Omega} h \left(\nu \cdot \bar{\gamma} D_i \left(\frac{A^i}{|A|} \right) - \bar{\gamma}^i \nu_j D_i \left(\frac{A^j}{|A|} \right) \right) \, d\mu
\end{aligned}$$

because $Du \cdot A \bar{\gamma} \cdot \nu = Du \cdot \bar{\gamma} A \cdot \nu$ and $A^j D_{ij} u \bar{\gamma}^i = \bar{\gamma}^j D_{ij} u A^i$. Now invoking (1.6) and (1.8) yields (1.7). \square

From the proof it is clear that the inequality in (1.5) need only hold at points of $\partial\Omega$ for which $h > 0$.

2. BOUNDS ON THE SOLUTION

In this section, we give pointwise estimates for solutions of (0.1). Since the proofs are the same as for the corresponding results in [12, §3] (see also [21, §2]) with Lemma 1.1 replacing [12, Lemma 2.1(a)], we only state the results.

Lemma 2.1. *Let $u \in C^{0,1}$ be a weak solution of (0.1). Suppose there are positive constants $m > 1$, a_1 , b_0 , b_1 , c_0 , M such that*

$$(2.1a) \quad p \cdot A(x, z, p) \geq |p|^m - a_1 |z|^m,$$

$$(2.1b) \quad zB(x, z, p) \leq b_0 p \cdot A(x, z, p) + b_1 |z|^m,$$

$$(2.1c) \quad z\varphi(x, z) \leq c_0 |z|^m \quad \text{for } x \in \partial\Omega$$

for $|z| \geq M$. Then, for any $q \geq 1$,

$$\begin{aligned}
(2.2) \quad \sup |u| &\leq C(m, n, q, \Omega) \\
&\quad \times \left(\left[(1 + a_1 + b_1 + c_0^{m/(m-1)})^{n/m} \int_{\Omega} |u|^q \, dx \right]^{1/q} + M \right). \quad \square
\end{aligned}$$

Lemma 2.2. *Let $u \in C^{0,1}$ be a weak solution of (0.1) and suppose there are positive constants $m > 1$, $m_0 \leq m$, a_1 , b_0 , b_1 , c_0 , M_0 such that*

$$(2.1a)' \quad p \cdot A(x, z, p) \geq |p|^m - a_1 |z|^{m_0},$$

$$(2.1b)' \quad zB(x, z, p) \leq b_0 p \cdot A(x, z, p) - b_1 |z|^{m_0},$$

$$(2.1c)' \quad z\varphi(x, z) \leq c_0|z|^{m_0} \quad \text{for } x \in \partial\Omega.$$

If $q \geq 1$ and if b_1 is sufficiently large (depending only on $a_1, b_0, c_0, q, m, \Omega$), then

$$(2.3) \quad \int_{\Omega} |u|^q dx \leq 4^{q/m_0} M_0^q |\Omega|. \quad \square$$

We emphasize that Lemmata 2.1 and 2.2 are valid for arbitrary Lipschitz domains. For the capillary problem, a bound on $z\varphi$ is not enough to guarantee a bound on u unless the bound on $z\varphi$ is tied into the geometry of Ω . The pseudonormal $\bar{\gamma}$ gives a natural connection.

Lemma 2.3. *Let $u \in C^{0,1}$ be a weak solution of (0.1) and suppose there are positive constants a_1, b_0, b_1, M such that (2.1a, b) hold with $m = 1$ and $|z| > M$. If there is $c_0 \in (0, 1)$ such that*

$$(2.1c)'' \quad z\phi(x, z) \leq c_0 \bar{\gamma} \cdot \gamma |z| \quad \text{for } x \in \partial\Omega, |z| \geq M,$$

then, for any $q \geq 1$

$$(2.4) \quad \sup |u| \leq C(c_0, n, q, \Omega) \left[\left((1 + a_1 + b_1)^n \int_{\Omega} |u|^q dx \right)^{1/q} + M \right]. \quad \square$$

Lemma 2.4. *Lemma 2.2 remains valid for $m = 1$ if (2.1c)'' replaces (2.1c)' and $c_0 < 1$. \square*

3. THE GRADIENT BOUND

We are now ready to derive gradient bounds for solutions of (0.1). In addition to the notation of §1, we use $\beta, c, \gamma_0, K, M, \varphi_0$ to denote nonnegative constants with

$$|u| \leq M \quad \text{and} \quad |\bar{\varphi}(x, u)| \leq \varphi_0 \quad \text{in } \Omega, \quad \gamma \cdot \bar{\gamma} \geq \gamma_0 > 0 \quad \text{on } \partial\Omega$$

(recall $\varphi = \bar{\varphi} \bar{\gamma} \cdot \gamma$ on $\partial\Omega$); β, c, K will often be subscripted as well.

We assume that there is a C^1 increasing function v_1 such that

$$(3.1) \quad 0 < \beta_0 v \leq v_1(F(x, u, Du) + \bar{\varphi} Du \cdot \bar{\gamma}) \leq v$$

in Ω and we also write v_1 for $v_1(F + \bar{\varphi} Du \cdot \bar{\gamma})$.

We define

$$\mathcal{E}^2 = \frac{1}{v^2} a^{ij} a^{km} D_{jk} u D_{im} u, \quad \mathcal{E} = \frac{1}{v^2 v_1'} a^{ij} D_i v_1 D_j v_1,$$

and for $\tau \geq \beta_0, x_0 \in \Omega$, and $\rho > 0$,

$$\Omega_{\tau} = \{x \in \Omega : v_1 > \tau\}, \quad \Gamma_{\tau} = \{x \in \partial\Omega : v_1 > \tau\},$$

$$B_{\rho} = \{x \in \mathbb{R}^n : |x - x_0| < \rho\}, \quad \Omega_{\tau, \rho} = B_{\rho} \cap \Omega_{\tau}, \quad \Gamma_{\tau, \rho} = B_{\rho} \cap \Gamma_{\tau}.$$

Next we introduce structure functions $w, \lambda, \lambda_0, \Lambda, \Lambda_0, \Lambda_1$, which are positive, increasing, and (except for Λ_0) C^1 on $[\tau_0, \infty)$ for some $\tau_0 \geq \beta_0$. These functions are related by the monotonicity conditions

$$(3.2a) \quad w^{\beta} (\Lambda/\lambda)^{(n+2)/2} / \Lambda \quad \text{is increasing,}$$

$$(3.2b) \quad w^{\beta} / \lambda_0 \quad \text{is increasing,}$$

(3.2c) $\xi^{-\beta}(\Lambda/\lambda)^{(n+2)/2}/\Lambda, \xi^{-\beta}w$ are decreasing
(for ξ the identity function), and

$$(3.3a) \quad \lambda_0^2 \leq \Lambda_0 \leq \Lambda,$$

$$(3.3b) \quad \lambda_0 \leq \Lambda_1$$

$$(3.3c) \quad (\xi\Lambda'_1)^2 + \Lambda_1^2 \leq \Lambda_0.$$

All of our conditions are assumed to hold on Ω_{τ_0} (or Γ_{τ_0} for (3.8d, f)). First, we assume (a^{ij}) to be elliptic on Ω_{τ_0} , so there are positive functions $\bar{\mu} \geq \underline{\mu}$ such that

$$(3.4) \quad \underline{\mu}|\zeta|^2 \leq a^{ij}\zeta_i\zeta_j \leq \bar{\mu}|\zeta|^2 \quad \text{for all } \zeta \in \mathbb{R}^n.$$

Moreover, we assume that

$$(3.5a) \quad \lambda_0 g^{ij}\zeta_i\zeta_j \leq a^{ij}\zeta_i\zeta_j,$$

$$(3.5b) \quad \lambda(1 + (v_1\lambda'/\lambda)^2)g^{ij}\zeta_i\zeta_j \leq \lambda_0 a^{ij}\zeta_i\zeta_j$$

for $\zeta \in \mathbb{R}^n$, and that

$$(3.5c) \quad \lambda_0 \leq |A + \overline{\varphi\gamma}| \leq \beta_1\lambda_0,$$

$$(3.5d) \quad \lambda_0 \leq \frac{1}{v_1'} \leq \beta_1\lambda_0,$$

$$(3.5e) \quad \lambda_0\bar{\mu} \leq \Lambda_0,$$

$$(3.5f) \quad \lambda\bar{\mu} \leq \lambda_0^3, \quad \lambda \leq \lambda_0\bar{\mu}.$$

Furthermore, we assume that

$$(3.6) \quad Du \cdot A \geq \beta_2\varphi_0v, \quad Du \cdot A \geq \beta_3|A|v$$

with $\beta_2 > 1$ and $\beta_3 > 0$.

Next, we suppose that there are functions C_k^i and D_k^i with D_k^i differentiable with respect to (x, z, p) such that

$$(3.7) \quad D_k u A_z^i + A_k^i + B\delta_k^i = C_k^i + D_k^i,$$

$$(3.8a) \quad \sum_{i,k} |C_k^i|^2 \leq \beta_4^2 \lambda_0 \Lambda_0 / \bar{\mu}, \quad \sum_k |C_k^i \nu_i|^2 \leq \beta_4^2 \underline{\mu} \Lambda_0 / \lambda_0,$$

$$(3.8b) \quad v^2 \sum_{i,j} |D_k^{ij}(A^k + \overline{\varphi\gamma^k})|^2 \leq \beta_4^2 \Lambda_0,$$

$$\sum_i |D_k^{ij} D_j u (A^k + \overline{\varphi\gamma^k})|^2 \leq \beta_4^2 \underline{\mu} \Lambda_0 / \lambda_0,$$

$$(3.8c) \quad (D_i u D_{kz}^i + D_{ki}^i)(A^k + \overline{\varphi\gamma^k}) \leq \beta_4^2 \Lambda_0$$

(here $D_k^{ij} = \partial D_k^i / \partial p_j$ and $D_{ki}^i = \partial D_k^i / \partial x^i$),

$$(3.8d) \quad -D_k^i \gamma_i (A^k + \overline{\varphi\gamma^k}) \leq \beta_4 \lambda_0 \Lambda_1,$$

$$(3.8e) \quad |\bar{\varphi}_x| \leq \beta_4 \lambda_0, \quad |D\bar{\gamma}| \leq \beta_4,$$

$$(3.8f) \quad \bar{\varphi}_z(Du \cdot A + \bar{\varphi} Du \cdot \bar{\gamma}) \leq \beta_4 \lambda_0 \Lambda_1,$$

$$(3.8g) \quad v^2 |A_x|^2 + |F_x|^2 \leq \beta_4^2 v^2 \lambda_0 \Lambda_0 / \bar{\mu},$$

$$(3.8h) \quad a^{ij} D_i u D_j u (|A_z|^2 + |\bar{\varphi}_z|^2 + v^{-2} F_z^2) \leq \beta_4^2 \lambda_0 \Lambda_0,$$

$$(3.9) \quad (\Lambda/\lambda)^{(n+2)/2} v \leq \beta_5 w^{\beta_6} Du \cdot A,$$

$$(3.10a) \quad v_1 (w')^2 \lambda_0^2 \leq \beta_7^2 \underline{\mu} Du \cdot A,$$

$$(3.10b) \quad \lambda_0 w \leq \beta_7 Du \cdot A,$$

$$(3.10c) \quad \bar{\mu} v \leq \beta_7^2 Du \cdot A,$$

$$(3.10d) \quad \Lambda_0 v_1 \leq \varepsilon(v_1) w^2 \lambda_0 Du \cdot A,$$

$$(3.10e) \quad v |A_z| + |A_x| \leq \beta_8 \varepsilon(v_1)^{1/2} Du \cdot A$$

for some positive decreasing function ε , and

$$(3.11a) \quad B \geq -\beta_9 Du \cdot A,$$

$$(3.11b) \quad |A| \leq \varepsilon_1(v) Du \cdot A \quad \text{for } v \geq \tau.$$

Let us point out that these conditions are essentially the same as in [12 and 21] (except that the latter work does not use the variational structure). In particular, our (3.10a) is the same as [21, (3.12a)] (in which the exponent 2 for (w') is missing) by virtue of [12, (4.11b)] and (3.1).

When $\partial\Omega$ is not smooth, we need some additional control of its geometry beyond being Lipschitz. Following [9], we assume also that it satisfies a uniform exterior sphere condition with radius R such that

$$(3.12) \quad R\beta_4 \geq 1,$$

and that $\partial\Omega$ is C^2 except for a singular set Γ of finite $(n-2)$ -dimensional Hausdorff measure. In particular, there is a sequence of C^1 functions (θ_k) such that

$$\theta_k = 0 \quad \text{on } \Gamma, \quad 0 \leq \theta_k \leq 1, \quad \|D\theta_k\|_{L^2} \leq K_0(\Gamma) \quad \text{for all } k,$$

and $\theta_k \rightarrow 1$ a.e. For $n > 2$, the existence of such a sequence follows from [24, Theorem 21 and 26, Lemma 8]. For $n = 2$, the sequence can be constructed easily because Γ consists of finitely many points. Furthermore, as we prove in the appendix, (3.12) implies

$$(3.12)' \quad D_i \gamma_j \zeta^i \zeta^j \leq |\zeta|^2 / R \quad \text{on } \partial\Omega \setminus \Gamma$$

for any vector ζ with $\zeta \cdot \gamma = 0$. In fact, we shall see in §7 that this condition, and not the size of the singular set, is what we use.

Lemma 3.1. *Let χ be a nonnegative Lipschitz function on $[\tau_0, \infty)$, let $\tau \geq \tau_0$ and suppose there is a constant c_χ such that*

$$(3.13) \quad 0 \leq (t - \tau)\chi'(t)/\chi(t) \leq c_\chi \quad \text{for } t \geq \tau.$$

If $\zeta \in C^{0,1}(\overline{\Omega})$ and if conditions (3.3a, c), (3.4), (3.5a, c, d, e), (3.6), (3.7), and (3.8) hold, then

$$(3.14) \quad \int_{\Omega_\tau} \left[\left(1 - \frac{\tau}{v_1}\right) \mathcal{E}^2 + \mathcal{E} \right] \chi \zeta^2 d\mu \\ \leq c_1(\beta_0, \beta_1, \beta_2, \beta_3, \gamma_0)(1 + c_\chi)^2 \int_{\Omega_\tau} [\beta_4^2 \Lambda_0 \zeta^2 + \lambda_0 a^{ij} D_i \zeta D_j \zeta] \chi d\mu.$$

Proof. If ζ vanishes on Γ , the estimate follows from the same arguments as in [12, Lemma 4.7; 21, Lemma 3.1] except for a slight modification concerning the boundary integrals. By using $\eta = (v_1 - \tau)_+ \chi(v_1) \zeta^2 (A + \overline{\varphi} \overline{\gamma})$ in [21, Lemma 1.1], we are led to examine the integral

$$I = \int_{\partial\Omega} [D_k \varphi + \varphi_z D_k u + A \cdot \delta_T \gamma_k - \gamma_i D_k^i] (A^k + \overline{\varphi} \overline{\gamma}^k) (v_1 - \tau)_+ \chi(v_1) \zeta^2 ds.$$

Since $D_i \gamma_k = D_k \gamma_i$,

$$\begin{aligned} D_k \varphi + \varphi_z D_k u &= D_k (\overline{\varphi} \gamma \cdot \overline{\gamma}) + \overline{\varphi}_z \gamma \cdot \overline{\gamma} D_k u \\ &= \gamma \cdot \overline{\gamma} [D_k \overline{\varphi} + \overline{\varphi}_z D_k u] + \overline{\gamma} \cdot D_k \gamma \overline{\varphi} + (D_k \overline{\gamma}) \cdot \gamma \overline{\varphi}, \end{aligned}$$

and $\gamma \cdot (A + \overline{\varphi} \overline{\gamma}) = 0$ on $\partial\Omega$, we have

$$\begin{aligned} I &= \int_{\partial\Omega} (A + \overline{\varphi} \overline{\gamma}) \cdot \delta_T \gamma_k (A^k + \overline{\varphi} \overline{\gamma}^k) (v_1 - \tau)_+ \chi(v_1) \zeta^2 ds \\ &\quad + \int_{\partial\Omega} [\gamma \cdot \overline{\gamma} \{D_k \overline{\varphi} + \overline{\varphi}_z D_k u\} + (\overline{\varphi} \gamma \cdot D_k \overline{\gamma})] (A^k + \overline{\varphi} \overline{\gamma}^k) (v_1 - \tau)_+ \chi(v_1) \zeta^2 ds \\ &\quad - \int_{\partial\Omega} \gamma_i D_k^i (A^k + \overline{\varphi} \overline{\gamma}^k) (v_1 - \tau)_+ \chi(v_1) \zeta^2 ds \\ &\leq c_2(n, \beta_1) \beta_4 \int_{\partial\Omega} \lambda_0 \Lambda_1 \left(1 - \frac{\tau}{v_1}\right)_+ \chi \zeta^2 v ds. \end{aligned}$$

Using Lemma 1.2 and [12, (4.11c)] in its more precise form

$$(3.15) \quad \lambda_0 a^{ij} D_{ik} u D_{jk} u \leq K_1(\beta_1, \beta_2, \beta_3) (\mathcal{E}^2 + \mathcal{E} + \beta_4^2 \Lambda_0) v^2$$

gives, for any $\eta > 0$,

$$\begin{aligned} I &\leq \eta \int_{\Omega_\tau} \left[\left(1 - \frac{\tau}{v_1}\right) \mathcal{E}^2 + \mathcal{E} \right] \chi d\mu \\ &\quad + c_3(n, \beta_0, \beta_1, \beta_2, \beta_3, \gamma_0) \frac{1}{\eta} (1 + c_\chi)^2 \int_{\Omega_\tau} [\beta_4^2 \Lambda_0 \zeta^2 + \lambda_0 a^{ij} D_i \zeta D_j \zeta] \chi d\mu \end{aligned}$$

just as on p. 238 of [12]. This estimate is just what is needed to prove (3.14) if ζ vanishes on Γ .

For general ζ , we first replace ζ by $\theta_k \zeta$ to obtain

$$\begin{aligned} & \int_{\Omega_\tau} \left[\left(1 - \frac{\tau}{v_1}\right) \mathcal{E}^2 + \mathcal{E} \right] \chi \theta_k^2 \zeta^2 d\mu \\ & \leq 2c_1(1 + c_\chi)^2 \int_{\Omega_\tau} [\beta_4^2 \Lambda_0 \zeta^2 + \lambda_0 a^{ij} D_i \zeta] \chi \theta_k^2 d\mu \\ & \quad + 2c_1(1 + c_\chi)^2 \int_{\Omega_\tau} \lambda_0 a^{ij} D_i \theta_k D_j \theta_k \chi \zeta^2 d\mu. \end{aligned}$$

By hypothesis, the right-hand side is bounded, independent of k , by some constant K_2 (which does depend on the gradient bound and on the specific function χ). Thus, for any compact subset S of $\bar{\Omega} \setminus \Gamma$, we have

$$\int_{\Omega_\tau} \left[\left(1 - \frac{\tau}{v_1}\right) \mathcal{E}^2 + \mathcal{E} \right] \chi I_s \theta_k^2 \zeta^2 d\mu \leq K_2,$$

where $I_s = 1$ on S and $= 0$ off of S . Sending $k \rightarrow \infty$ allows us to replace θ_k by 1 in this inequality. Then choosing a sequence of S 's which expand to Ω yields

$$\int_{\Omega_\tau} \left[\left(1 - \frac{\tau}{v_1}\right) \mathcal{E}^2 + \mathcal{E} \right] \chi \zeta^2 d\mu \leq K_2.$$

Since this integral is finite, the proof of (3.14) given for ζ vanishing on Γ applies for arbitrary ζ . \square

Before proceeding to our gradient estimate, we prove a Sobolev inequality which involves several of our structure conditions. We remark that a much simpler form of this inequality can be used if $\partial\Omega \in C^2$ (see [12, Lemmata 1.3 and 2.1(b)]).

Lemma 3.2. *Let $h \in W^{1,2} \cap L^\infty$ be nonnegative and suppose $h = 0$ anywhere that $v_1 < \tau_0$. Suppose also that structure conditions (3.4), (3.5b, c, e, f), (3.6), (3.8e, g, h) are satisfied. Then*

$$\begin{aligned} (3.16) \quad \int_{\Omega} h^{2(n+2)/n} d\mu & \leq c_4(\beta_0, \beta_1, \beta_2, \beta_3, \gamma_0, n) \left(\int_{\Omega} h^2 d\mu \right)^{2/n} \\ & \quad \times \left(\int_{\Omega} [|\delta h|^2 + h^2(\mathcal{E}^2 + \mathcal{E} + \beta_4^2 \Lambda_0)/\lambda] d\mu \right). \end{aligned}$$

Proof. As in [12, Lemma 1.3], it follows from [25, p. 372; 13, Lemma 1.1] that if the nonnegative numbers p, q, r satisfy $\frac{p}{n} + r + 1 = q$, then

$$\begin{aligned} \int_{\Omega} h^q d\mu & \leq c_5(p, q, r) \left(\int_{\Omega} h^p d\mu \right)^{1/2} \\ & \quad \times \left(\int_{\Omega} \left[h^r |\delta h| + h^{r+1} \left| \frac{1}{nv} g^{ij} D_{ij} u \right| \right] d\mu + \int_{\partial\Omega} h^{r+1} v ds \right). \end{aligned}$$

Now we use Lemma 1.2 with h replaced by h^{r+1} to estimate the boundary

integral. We have

$$\begin{aligned}
& Du \cdot \bar{\gamma} D_i \left(\frac{A^i}{|A|} \right) - \bar{\gamma}^i D_j u D_i \left(\frac{A^j}{|A|} \right) \\
&= \frac{1}{|A|} a^{ik} D_{jk} u (\delta_i^j \nu \cdot \bar{\gamma} - \bar{\gamma}^j \nu_i) - \frac{1}{|A|^3} a^{ik} D_{jk} u (\nu \cdot A A^i \bar{\gamma}^j - \nu \cdot \bar{\gamma} A^i A^j) \\
&\quad + \frac{1}{|A|} (\bar{\gamma}^i D_i u A_j^j - \bar{\gamma}^j D_i u A_j^i) - \frac{1}{|A|^3} (Du \cdot \bar{\gamma} A^k A_j^k A^j - Du \cdot A \bar{\gamma}^j A^k A_j^k) \\
&\leq c_6(n, \beta_1, \beta_2, \beta_3) \left[(a^{ij} D_{ik} u D_{jk} u)^{1/2} \left(\frac{\bar{\mu}}{\lambda_0^2} \right)^{1/2} + \beta_4 \left(\frac{\Lambda_0}{\bar{\mu} \lambda} \right)^{1/2} v \right] \\
&\leq c_7(n, \beta_1, \beta_2, \beta_3) [(\mathcal{E}^2 + \mathcal{E} + \beta_4^2 \Lambda_0)/\lambda]^{1/2} v
\end{aligned}$$

by (3.4), (3.5c), and (3.8e, g, h), (3.15). Hence Lemma 1.2 implies that

$$\begin{aligned}
\int_{\partial\Omega} h^{r+1} v \, ds &\leq c_8(\beta_3, \gamma_0) \int_{\partial\Omega} h^{r+1} Du \cdot \frac{A}{|A|} \gamma \cdot \bar{\gamma} \, ds \\
&\leq c_9(\beta_1, \beta_2, \beta_3, \gamma_0, n) \int_{\Omega} \{r h^r |\delta h| + h^{r+1} [(\mathcal{E}^2 + \mathcal{E} + \beta_4^2 \Lambda_0)/\lambda]^{1/2}\} \, d\mu.
\end{aligned}$$

Observing from (3.5b) that

$$\lambda \left| \frac{1}{nv} g^{ij} D_{ij} u \right|^2 \leq \frac{\lambda_0}{v^2} (a^{ij} D_{ik} u D_{jk} u),$$

we see that

$$\begin{aligned}
\int_{\Omega} h^q \, d\mu &\leq c_{10}(\beta_1, \beta_2, \beta_3, \gamma_0, n, p, r) \left(\int_{\Omega} h^p \, d\mu \right)^{1/n} \\
&\quad \times \int_{\Omega} [h^r |\delta h| + h^{r+1} \{(\mathcal{E}^2 + \mathcal{E} + \beta_4^2 \Lambda_0)/\lambda\}^{1/2}] \, d\mu.
\end{aligned}$$

This inequality is just [13, (1.3)] with $|H|$ replaced by $\{(\mathcal{E}^2 + \mathcal{E} + \beta_4^2 \Lambda_0)/\lambda\}^{1/2}$, so the proof is completed by choosing $q = 2(n+2)/n$, $p = 1 + q/2$, and $r = 1 + 2/n - 2/n^2$, and then applying Schwarz's inequality and Hölder's inequality as on [13, p. 581]. \square

From this point, the proof of our gradient bound proceeds as in [12] with the obvious changes (for example, Lemma 3.2 in place of [12, Lemma 1.3] and $\sup_{B_\rho \cap \Omega} u - u$ in place of $u - \inf u$ in the proof of Lemma 3.5), the sharper form of the calculations in [21], and the observation that our (3.3a), and (3.5b, e, f) can be used in place of [12, (4.11e)]. For this reason we only state the results.

Lemma 3.3. *Suppose conditions (3.2a, c) and (3.3)–(3.8) are satisfied. Then for any $x_0 \in \Omega$, $\tau \geq \tau_0$, and $\rho > 0$, we have*

$$\begin{aligned}
(3.17) \quad & \sup_{\Omega_{\tau, \rho/2}} \left\{ \left(1 - \frac{\tau}{v_1} \right)^{n+2} w^2 \right\} \\
& \leq c_{11}(\beta, \beta_1, \beta_2, \beta_3, \beta_4 \rho, \gamma_0, n) \rho^{-n} \int_{\Omega_{\tau, \rho}} w^2 \left(\frac{\Lambda}{\lambda} \right)^{(n+2)/2} v \, dx. \quad \square
\end{aligned}$$

Lemma 3.4. *Suppose conditions (3.2b, c) (3.3a, c) (3.4), (3.5a, c, d, e), (3.6), (3.7), (3.8), and (3.10) are satisfied, let $q > 0$, and set*

$$(3.18) \quad \sigma = \operatorname{osc}_{\Omega \cap B_\rho} u.$$

Then there is a constant $c_{12} = c_{12}(\beta, \beta_1, \beta_2, \beta_3, \beta_7, \gamma_0, n, q)$ such that if there is a $\tau_1 \geq \tau$ for which

$$(3.19) \quad 8(c_{12}\beta_4^2 + \beta_8^2)\sigma^2\varepsilon(\tau_1) \leq 1,$$

then

$$(3.20) \quad \int_{\Omega_{\tau, \rho/2}} w^q Du \cdot A dx \leq c_{13}(\beta, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4\rho, \beta_7, \beta_8\rho, \gamma_0, n, q) \\ \times \left[w(\tau_1) + \frac{\sigma}{\rho} \right]^q \int_{\Omega_{\tau, \rho}} Du \cdot A dx. \quad \square$$

For our next lemma, we introduce the quantities

$$\Omega'_\tau = \{x \in \Omega : v(x) > \tau\}, \quad \Omega'_{\tau, \rho} = \Omega'_\tau \cap B_\rho, \quad K(\rho) = \rho^{1-n}|B_\rho \cap \partial\Omega|.$$

Lemma 3.5. *Suppose conditions (3.12a, b) hold on Ω'_τ and define σ by (3.18). Then for any $\rho > 0$, $x_0 \in \Omega$, and $\tau_2 \geq \tau$ such that*

$$(3.21a) \quad \varepsilon_1(\tau_2) \leq \frac{\rho}{8\sigma}$$

and

$$(3.21b) \quad \Delta_1 = \sup_{\substack{x \in B_\rho \\ v \leq \tau_2}} \left\{ \sigma(B + \beta_9 Du \cdot A)_- + (Du \cdot A)_- + \frac{4\sigma|A|}{\rho} \right\},$$

we have

$$(3.22) \quad \int_{\Omega'_{\tau, \rho/2}} Du \cdot A dx \leq c_{14}(n) \exp(\beta_9\sigma)\rho^n \left(K(\rho)\frac{\sigma}{\rho} + \Delta_1 \right). \quad \square$$

Although Lemmata 3.3–3.5 are readily combined to give a gradient bound, we shall not do so here. The details are generally straightforward but, as we shall see in §5, it is sometimes advantageous to use only the estimates from Lemmata 3.3 and 3.5.

As in [28, §3] (and [21, §4]), the form of our estimates can be improved if the structure conditions of §3 are appropriately strengthened. Since these estimates are proved by modifying the proofs of §3 in exactly the way discussed in [28, 21], we state them without proof.

Theorem 3.6. *Let $x_0 \in \Omega$, $\rho > 0$, and suppose conditions (3.1)–(3.10) hold with $\varepsilon(v_1) = \beta_{10}w^{-2}$. Suppose also that*

$$(3.23) \quad B \geq -\beta_{11}|A| \quad \text{in } \Omega'_{\tau_0}.$$

With σ given by (3.18), set

$$(3.24a) \quad \tau_3 = \max \left\{ \tau_0, \frac{8\sigma}{\beta_3\rho} + 2\frac{\beta_{11}\sigma}{\beta_3} \right\},$$

$$(3.24b) \quad \Delta_2 = \sup_{|Du| \leq \tau_3} \left\{ \sigma B_- + (Du \cdot A)_- + \frac{\sigma}{\rho}|A| \right\}.$$

Then there is a constant $c_{15} = c_{15}(\beta, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4\rho, \beta_5, \beta_7, \beta_8\rho, \beta_{10}, \beta_{11}\rho, \gamma_0, K(\rho), n)$ such that

$$(3.25) \quad \sup_{\Omega_{\tau, \rho/8}} \left(1 - \frac{\tau_0}{v_1}\right)_+^{n+2} w^2 \leq c_{15} \left[w(\tau_0) + \frac{\sigma}{\rho}\right]^{2+\beta_6} \left(\Delta_2 + \varphi_0 \frac{\sigma}{\rho}\right).$$

If also

$$(3.26) \quad a^{ij} \xi_i \eta_j \leq \beta_{12} \{\Lambda_0 g^{ij} \eta_i \eta_j\}^{1/2} \{a^{ij} \xi_i \xi_j\}^{1/2}$$

for all η, ξ in \mathbb{R}^n , set

$$(3.27a) \quad \sigma_0 = \sup_{\Omega \cap B_\rho} (u - u(x_0))_+,$$

$$(3.27b) \quad \tau_4 = \max \left\{ \tau_0, \frac{8\sigma_0}{\beta_3\rho} + \frac{2\beta_{11}}{\beta_3} \sigma_0 \right\},$$

$$(3.27c) \quad \Delta_3 = \sup_{\substack{|Du| \leq \tau_4 \\ u - u(x_0) \geq \rho}} \left\{ \sigma_0 B_- + (Du \cdot A)_- + \frac{\sigma_0}{\rho} |A| \right\}.$$

Then there is $c_{16} = c_{16}(\beta, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4\rho, \beta_5, \beta_7, \beta_8\rho, \beta_{10}, \beta_{11}\rho, \beta_{12}, \gamma_0, K(\rho), n)$ such that

$$(3.28) \quad \sup_{\Omega_{\tau, \rho/8}} \left(1 - \frac{\tau_0}{v_1}\right)_+^{n+2} w^2 \leq c_{16} [1 + w(\tau_0)] \left[1 + w(\tau_0) + \frac{\sigma}{\rho}\right]^{1+\beta_6} \left(\Delta_3 + \varphi_0 \frac{\sigma}{\rho}\right). \quad \square$$

4. THE PARABOLIC CASE

Parabolic versions of the results in §3 are also valid. We only state analogs of Lemmata 3.3, 3.4, and 3.5 including a version which allows localization in time as well as space. Refined estimates are then easy to obtain (see [21, §4]).

To state these estimates, we introduce some additional terminology: For $T > 0$, we set

$$Q_T = \Omega \times (0, T), \quad SQ_T = \partial\Omega \times (0, T),$$

and we consider the problem

$$\begin{aligned} \operatorname{div} A(x, t, u, Du) + B(x, t, u, Du) &= u_t \quad \text{in } Q_T, \\ A(x, t, u, Du) \cdot \gamma + \varphi(x, t, u) &= 0 \quad \text{on } SQ_T, \\ u(\cdot, 0) &= u_0 \quad \text{in } \Omega \end{aligned}$$

for some known function u_0 . (Here Du is only the spatial gradient of u .) Next we define

$$\begin{aligned} Q(\tau) &= \{(x, t) \in Q_T : v_1(x, t) > \tau\}, \\ Q(\tau, \rho) &= \{(x, t) \in Q(\tau) : x \in B_\rho\}, \\ SQ(\tau) &= \{(x, t) \in SQ_T : v_1(x, t) > \tau\}. \end{aligned}$$

Moreover all structure conditions are assumed to hold with $Q(\tau_0)$ replacing Ω_{τ_0} and $SQ(\tau_0)$ replacing Γ_{τ_0} .

Since A , φ , and F may depend on t , we assume the following bounds:

$$(4.1a) \quad \overline{\varphi}_t \overline{\gamma} \cdot Du + F_t \leq \beta_4^2 \Lambda_0,$$

$$(4.1b) \quad \overline{\mu} F_z^2 \leq \beta_4^2 \lambda_0 \Lambda_0,$$

$$(4.1c) \quad B F_z \leq \beta_4^2 \Lambda_0,$$

$$(4.1d) \quad (v|A_z| + |A_x|)|F_z| \leq \beta_4^2 \Lambda_0,$$

$$(4.2) \quad \lambda \leq \lambda_0 v,$$

and we suppose that

$$(4.3) \quad \xi^{-\beta} \lambda_0 \text{ is decreasing.}$$

Lemma 4.1. *Let χ be a nonnegative Lipschitz function on $[\tau_0, \infty)$, let $\tau \leq \tau_0$, and suppose there is a constant c_χ for which (3.14) holds. If $\zeta \in C^{0,1}(Q_T)$, if $\zeta(x, 0) = 0$ in Ω or if $v_1(x, 0) \leq \tau$ in Ω , and if conditions (3.3a, c), (3.4), (3.5a, c, d, e), (3.6), (3.7), (3.8), (4.1), and (4.3) hold, then there is a constant $c_{17} = c_{17}(\beta, \beta_0, \beta_1, \beta_2, \beta_3, \gamma_0, n)$ such that*

$$(4.4) \quad \sup_{0 < t < T} \int_{\Omega \times \{t\}} (v_1 - \tau)_+^2 \lambda_0 \chi \zeta^2 dx + \int_0^T \int_{\Omega_\tau} \left[\left(1 - \frac{\tau}{v_1}\right) \mathcal{E}^2 + \mathcal{E} \right] \chi \zeta^2 d\mu dt \\ \leq c_{17}(1 + c_\chi)^3 \left\{ \int_0^t \int_{\Omega_\tau} [\beta_4^2 \Lambda_0 \zeta^2 + \lambda_0 a^{ij} D_i \zeta D_j \zeta] \chi d\mu dt \right. \\ \left. + \int_0^T \int_{\Omega_\tau} (v_1 - \tau)^2 \chi |\zeta_t| \zeta dx dt \right\}.$$

Proof. From the proof of Lemma 3.1, we find that

$$\int_0^h \int_{\Omega_\tau} \left[\left(1 - \frac{\tau}{v_1}\right) \mathcal{E}^2 + \mathcal{E} \right] \chi \zeta^2 d\mu dt \\ \leq c_1(1 + c_\chi)^2 \int_0^h \int_{\Omega_\tau} [\beta_4^2 \Lambda \zeta^2 + \lambda_0 a^{ij} D_i \zeta D_j \zeta] \chi d\mu dt \\ + c_{18}(c_1) \int_0^h \int_{\Omega} u_t D_k [(A^k + \overline{\varphi} \overline{\gamma}^k)(v_1 - \tau) \chi \zeta^2] dx dt$$

for any $h \in (0, T)$. Thus it suffices to estimate this last integral, which we denote by I .

Setting

$$\Xi(s) = \int_\tau^s \frac{(\sigma - \tau)_+ \chi(\sigma)}{v'_1(v_1^{-1}(\sigma))} d\sigma$$

and using the approximation and integration by parts argument from [13, Lemma 3.1] yields

$$(4.5) \quad I = - \int_{\Omega \times \{h\}} \Xi(v_1) \zeta^2 dx + 2 \int_0^h \int_{\Omega_\tau} \Xi(v_1) \zeta \zeta_t dx dt \\ + \int_0^h \int_{\Omega_\tau} u_t F_z (v_1 - \tau) \chi \zeta^2 dx dt \\ + \int_0^h \int_{\Omega_\tau} (F_t + \overline{\varphi}_t \overline{\gamma} \cdot Du) (v_1 - \tau) \chi \zeta^2 dx dt.$$

By using [13, (3.6)] with $\chi\lambda_0$ replacing χ , we see from (4.3) and (3.5d) that

$$\frac{\beta_1}{2}\lambda_0\chi(v_1 - \tau)^2 \geq \Xi \geq \frac{1}{2(1 + \beta + c_\chi)}\lambda_0\chi(v_1 - \tau)^2.$$

We use this inequality to estimate the first two integrals in (4.5). In the third integral, we use the differential equation in the form

$$u_t = \frac{1}{v}a^{ij}D_{ij}u + B + A_z^iD_iu + A_i^i.$$

Thus

$$\begin{aligned} I \leq & -\frac{1}{2(1 + \beta + c_\chi)} \int_{\Omega \times \{h\}} (v_1 - \tau)^2 \lambda_0 \chi \zeta^2 dx + \beta_0 \int_0^h \int_{\Omega_\tau} (v_1 - \tau)^2 \chi |\zeta_t| \zeta dx dt \\ & + \int_0^h \int_{\Omega_\tau} \left(1 - \frac{\tau}{v_1}\right) \frac{\lambda_0}{v^2} a^{ij} D_{ik} u D_{jk} u]^{1/2} \left[\frac{\bar{\mu}}{\lambda_0} F_z^2\right]^{1/2} \chi \zeta^2 d\mu dt \\ & + \int_0^h \int_{\Omega_\tau} [BF_z + (v|A_z| + |A_x|)F_z] \chi \zeta^2 d\mu dt \\ & + \int_0^h \int_{\Omega_\tau} (F_t + \bar{\varphi}_t \bar{\gamma} \cdot Du) \chi \zeta^2 d\mu dt. \end{aligned}$$

If we estimate these last three integrals via (4.1), Schwarz's inequality, and (3.16), we obtain (4.4). \square

Using Lemma 4.1 in place of Lemma 3.1 gives analogs of Lemmata 3.3, 3.4, 3.5 by virtue of the proofs of [21, Lemmata 3.2, 3.3, 3.4, Theorem 4.1].

Lemma 4.2. *Suppose conditions (3.2a, c), (3.3)–(3.8), (4.1), (4.2), and (4.3) are satisfied, and let $x_0 \in \Omega$, and $\rho > 0$.*

(a) *If $v_1 \leq \tau_0$ on $\Omega \times \{0\}$, then*

$$(4.6) \quad \sup_{Q(\tau, \rho/2)} \left(1 - \frac{\tau}{v_1}\right)^{n+2} w^2 \leq c_{19}(\beta, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4 \rho, \gamma_0, n) \\ \times \rho^{-n-2} \iint_{Q(\tau, \rho)} w^2 \left(\frac{\Lambda}{\lambda}\right)^{(n+2)/2} v dx dt.$$

(b) *If also*

$$(4.7) \quad \lambda_0 v \leq \beta_{13} \Lambda,$$

and $\rho^2 < t_0 < T$, then

$$(4.8) \quad \left(1 - \frac{\tau}{v_1}\right)_+^{n+2} w^2(x_0, t_0) \leq c_{20}(\beta, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4 \rho, \beta_{13}, \gamma_0, n) \\ \times \rho^{-n-2} \int_{t_0 - \rho^2}^{t_0} \int_{\Omega_{\tau, \rho}} w^2 \left(\frac{\Lambda}{\lambda}\right)^{(n+2)/2} v dx dt. \quad \square$$

Lemma 4.3. *Suppose conditions (3.2b, c), (3.3a, c), (3.4), (3.5a, c, d, e), (3.6), (3.7), (3.8), (3.10), (4.1), (4.2), and (4.3) are satisfied, and let $q > 0$, $x_0 \in \Omega$, and $\rho > 0$.*

(a) If $v_1 \leq \tau$ on $\Omega \times \{0\}$, set

$$(4.9) \quad \sigma = \sup\{|u(x, t) - u(x_0, t)| : x \in B_\rho, 0 < t < T\}.$$

Then there is a constant $c_{21} = c_{21}(\beta, \beta_1, \beta_2, \beta_3, \beta_7, \gamma_0, n, q)$ such that if there is a $\tau_1 \geq \tau$ for which

$$(4.10) \quad 8(c_{21}\beta_4^2 + \beta_8^2)\sigma^2\varepsilon(\tau_1) \leq 1,$$

then

$$(4.11) \quad \begin{aligned} & \iint_{Q(\tau, \rho/2)} w^q Du \cdot A dx dt \\ & \leq c_{22}(\beta, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4\rho, \beta_7, \beta_8\rho, \gamma_0, n, q) \\ & \quad \times \left[w(\tau_1) + \frac{\sigma}{\rho} \right]^q \iint_{Q(\tau, \rho)} Du \cdot A dx dt. \end{aligned}$$

(b) If also

$$(4.12) \quad v^2 \leq \beta_{14} w Du \cdot A$$

and $\rho^2 < t_0 < T$, set

$$(4.13) \quad \sigma = \sup\{|u(x, t) - u(x_0, t)| : x \in B_\rho, t_0 - \rho^2 < t < t_0\}.$$

If there is a $\tau_1 \geq \tau$ for which (4.10) holds, then there is a constant $c_{23} = c_{23}(\beta, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4\rho, \beta_7, \beta_8\rho, \gamma_0, n, q)$ such that

$$(4.14) \quad \begin{aligned} & \int_{t_0 - \rho^2/4}^{t_0} \int_{\Omega_{\tau, \rho/2}} w^q Du \cdot A dx dt \\ & \leq c_{23} \left[w(\tau_1) + \frac{\sigma}{\rho} + \beta_{14} \left(\frac{\sigma}{\rho} \right)^2 \right] \int_{t_0 - \rho^2}^{t_0} \int_{\Omega_{\tau, \rho}} Du \cdot A dx dt. \quad \square \end{aligned}$$

For our final lemma, we define

$$Q'(\rho) = \{(x, t) \in Q_T : x \in B_\rho, t_0 - \rho^2 < t < t_0\},$$

$$Q'(\rho, \tau) = \{(x, t) \in Q'(\rho) : v(x, t) \geq \tau\}.$$

Lemma 4.4. Let $\tau \geq 1$, $\rho > 0$, $x_0 \in \Omega$, and $0 < t_0 < T$. Suppose conditions (3.11) hold on $Q'(\rho, \tau)$ and set

$$(4.15) \quad M = \operatorname{osc}_{Q'(\rho)} u.$$

If $\tau_5 \geq \tau$ is so large that

$$(4.16) \quad \varepsilon_1(\tau_5) \leq \frac{\rho}{8M},$$

and if

$$(4.17) \quad \Delta_4 = \sup_{Q'(\rho) \setminus Q'(\rho, \tau)} \left\{ M(B + \beta_9 Du \cdot A)_- + (Du \cdot A)_- + \frac{M}{\rho} |A| \right\},$$

then

$$(4.18) \quad \iint_{Q'(\rho/2, \tau)} Du \cdot A dx dt \leq c_{24}(n) \exp(\beta_9 M) \rho^n [M^2 + \Delta_4 \rho^2 + K(\rho) M \rho \varphi_0]. \quad \square$$

Note that Lemma 4.4 is identical to [21, Lemma 3.4].

5. EXAMPLES

Our structure conditions, which are based on those in [28], were chosen to include all the examples in [21] and hence all those in [28, 12, 30] as well. Rather than reproduce them all here, we only include three to show the range of these methods.

Example 1 (The capillary problem). Suppose

$$A = \nu, \quad B = nH(x, u, Du) + b(x, u, Du)$$

for Lipschitz H with

$$\begin{aligned} nH &\geq -\theta_1, & v|b| &\leq \theta_1, \\ nv[|H_p| + |H_p \cdot p|] &\leq \theta_1, & n[vH_z + |H_x|] &\leq \theta_1^2, \\ \varphi_0 &< 1, & \varphi_z &\leq 0. \end{aligned}$$

We then set

$$\begin{aligned} C_k^i &= b\delta_k^i, \quad D_k^i = nH\delta_k^i, \quad v_1 = \frac{1}{2}(v + \varphi Du \cdot \gamma), \\ \underline{\mu} &= v^{-2}, \quad w = \log v_1, \quad \lambda^{1/3} = \lambda_0 = \frac{1}{2}(1 - \varphi_0), \quad \bar{\mu} = \Lambda = \Lambda_0 = \Lambda_1 = 1. \end{aligned}$$

With these choices,

$$\begin{aligned} \beta &= 1, \quad \beta_4 = c(\varphi_0) \left(\theta_1 + \frac{1}{R} \right), \quad \beta_6 = 0, \quad \beta_7 = 2, \\ \beta_8 &= 0, \quad \beta_{11} = c(\varphi_0)\theta_1, \end{aligned}$$

and $\tau_0, \beta_0, \dots, \beta_3, \beta_5, \beta_{10}$ chosen suitably (depending only on φ_0 and n), Theorem 3.6 gives an estimate of the form

$$v(x_0) \leq c_1 \exp(c_2 \sigma_0 / \rho)$$

with c_1 and c_2 depending only on $\theta_1 \rho, K(\rho), n, \gamma_0, \varphi_0$, and ρ/R . (Recall that R is the radius of the exterior sphere.) Because of the restrictions on φ forced by the inequality $\varphi_0 < 1$, this estimate is a sharp local version of [9, Lemma 4.5].

Let us examine briefly the special case that Γ is an edge from [9, Theorem 4.2]. Then, at least locally, Γ is the intersection of two C^2 hypersurfaces S_1 and S_2 with inner normals γ_1 and γ_2 . Let θ denote the angle between S_1 and S_2 along Γ ; by the exterior sphere condition $\theta < \pi$. If $\varphi = \cos \beta$ is Lipschitz with $|\pi - 2\beta| < \theta$ on Γ , then an appropriate pseudonormal is $\bar{\gamma} = \frac{(\gamma_1 + \gamma_2)}{|\gamma_1 + \gamma_2|}$ (cf. [9, p. 28]). More generally we obtain a gradient bound if the restrictions φ_1 and φ_2 of φ to S_1 and S_2 , respectively, are Lipschitz and if there are functions θ_1 , and θ_2 such that

$$\pi > \theta = \theta_1 + \theta_2, \quad \sin \theta_i > |\varphi_i|, \quad \varphi_1 \varphi_2 > 0 \text{ or } \varphi_1 = \varphi_2 = 0 \text{ on } \Gamma,$$

so that a suitable $\bar{\gamma}$ can be constructed. The restriction that φ not change sign on Γ is clearly needed for the pseudonormal approach to work, but it is not mentioned in [9]. On the other hand, this restriction is not needed to conclude the Hölder continuity of the gradient for Lipschitz solutions as in [20]. Remark 4.8 from [9] is also applicable; it is generally not possible to construct a pseudonormal near singular sets of higher codimension.

Example 2 (Uniformly elliptic problems). Here we suppose there are nonnegative functions ε and ψ , with ε decreasing, $\varepsilon\xi$ increasing, $\varepsilon \rightarrow 0$ at infinity, $\psi \in C^1(1, \infty)$, and $\psi(t) > 0$ and $0 \leq t\psi'(t)/\psi(t) \leq \theta_0$ for $t > 1$ such that

$$(5.1a) \quad Du \cdot A \geq v\psi(v) - \theta_1, \quad |A| \leq \theta_2\psi(v),$$

$$(5.1b) \quad \psi(v)|\xi|^2 \leq a^{ij}\xi_i\xi_j \leq \theta_3\psi(v)|\xi|^2,$$

$$(5.1c) \quad |F_z| + v^{-1}|F_x| + v|A_z| + |A_x| + |B| \leq \theta_4\varepsilon(v)^{1/2}v\psi(v), \quad |\bar{\varphi}_x|, |\bar{\varphi}_z| \leq \theta_4.$$

Let us also assume for simplicity that $\psi \rightarrow \infty$ at infinity and that $\varepsilon\psi^2$ is bounded away from zero. Now set $I(t) = \int_1^t \psi(s) ds$, and choose

$$\begin{aligned} F + \varphi Du \cdot \gamma &= 2I(v_1), \quad D_k^i = 0, \\ \underline{\mu} &= \psi(v), \quad w = v_1, \quad \lambda_0 = \frac{1}{2}\psi(v_1), \quad \lambda = \frac{1}{2}, \quad \Lambda_0 = \frac{1}{2}\varepsilon(v_1)(v_1\psi(v_1))^2, \\ \bar{\mu} &= \theta_3\psi(v), \quad \Lambda = (v_1\psi(v_1))^2, \quad \Lambda_1 = c(\theta_0)(v_1 + \psi(v_1)). \end{aligned}$$

For $\beta_4 = c(n)(\theta_4 + \frac{1}{R} + |D\bar{\gamma}|)$, $\tau_0, \beta, \beta_0, \dots, \beta_3, \beta_5, \beta_6, \beta_7, \beta_9$ depending on $\theta_0, \dots, \theta_3$, and $\beta_8 = c(\theta_0, \dots, \theta_3)\theta_4$, we can combine Lemmata 3.3, 3.4, and 3.5 to obtain a gradient bound of the form

$$v(x_0) \leq c \left(\theta_0, \theta_1, \theta_2, \theta_3, \theta_4\rho, \frac{\rho}{R}, \rho|D\bar{\gamma}|, \gamma_0, n, \varphi_0, \frac{\sigma}{\rho} \right).$$

This estimate is also valid for degenerate equations, in which case the first inequality in (5.1c) can be replaced by

$$|F_z| + v^{-1}|F_x| + v|A_z| + |A_x| + |B| \leq \theta_4\varepsilon(v)^{1/2}v\psi(v) + \theta_5.$$

For the nondegenerate case we have a modulus of continuity estimate from [18] (with appropriate geometry) and if ψ and $v\psi$ are replaced by $(|Du| + \delta)^m$ and $(|Du| + 1)^{m+1}$, respectively, for some $m > 0$ and $\delta \in [0, 1]$, then a modulus of continuity estimate follows from [10, Theorem 10.2.1]. In both cases, this modulus can be obtained (independent of δ in the second case) even for $\varepsilon = 1$. By taking ρ sufficiently small, we obtain a gradient bound in this case as well.¹

Note that now, near an edge, the modulus of continuity estimates of [18] apply, so the only restrictions on φ_1 and φ_2 are that

$$(5.2) \quad \varphi_1\varphi_2 > 0 \text{ or } \varphi_1 = \varphi_2 = 0 \quad \text{on } \Gamma.$$

If also $\psi(v) = v$, [19, Theorem 4.1] shows that u has Hölder continuous gradient without the present gradient bounds and hence without the variational structure and without the restriction (5.2).

Example 3. Set $\Psi(v) = \exp(\theta v^2)$, and suppose

$$A = \Psi(v)p, \quad |B| \leq \theta_1 v^2 \Psi(v), \quad |\bar{\varphi}_x| + |\bar{\varphi}_z| \leq \theta_1.$$

Then we define v_1 by

$$\frac{1}{2\theta}\Psi(v) + \bar{\varphi} Du \cdot \bar{\gamma} = \frac{1}{2\theta}\Psi(v_1),$$

¹Note added in proof. If $\delta = t\psi'(t)\psi(t) \leq \theta_0$ for some $\delta > 0$, then (5.1a) and $|B| \leq \theta_4 v\psi(v) + \theta_5$ imply a modulus of continuity by the arguments in [31], so we can take $\varepsilon = 1$ in this case also.

and observe that $|v^2 - v_1^2|$ is bounded and goes to zero as $v, v_1 \rightarrow \infty$. Now we choose

$$\begin{aligned} w &= (v_1)^{1/2}, \quad \underline{\mu} = \lambda_0 = v_1 \Psi(v_1), \quad \bar{\mu} = (1 + \theta v^2) v \Psi(v), \\ \lambda &= v_1^5 \exp\left(\frac{n\theta}{n+2} v_1^2\right), \quad \Lambda = \Lambda_0 = (1 + \theta) v_1^5 \Psi(v_1), \\ \Gamma_1 &= \theta_1 v_1 \Psi(v_1), \quad c_k^i = B \delta_k^i \end{aligned}$$

to infer that (3.2a, c), (3.3)–(3.8), and (3.12a) are satisfied with

$$\beta = 5, \quad \beta_4 = c(n, \theta) \left(\theta_1 + \frac{1}{R} + |D\bar{\gamma}|_0 \right)$$

and $\beta_0, \dots, \beta_3, \beta_9$ depending only on n and θ provided $\tau_0 = c(\theta, \varphi_0)$. Thus Lemmata 3.3 and 3.5 along with the inequality

$$v_1 \left(\frac{\Lambda}{\lambda} \right)^{(n+2)/2} v \leq c(\theta, n, \varphi_0) Du \cdot A$$

imply that

$$v(x_0) \leq c(n, \theta, \varphi_0, \theta_1 \rho, |D\bar{\gamma}|_0 \rho, K(\rho), \gamma_0) \Psi\left(\frac{\sigma}{\rho}\right) \frac{\sigma}{\rho}.$$

Note that, near an edge, we need (5.2) to hold just as in Example 2.

We close by pointing out (see, e.g., [12, p. 250]) that we can consider a situation intermediate to the given examples, namely $A = \Psi(v)p$ for some positive $C^1([1, \infty))$ function Ψ satisfying

$$-1 \leq \frac{t\Psi'(t)}{\Psi(t)} \leq \theta,$$

for $\theta > -1$ a constant and suitable conditions on B .

6. OBSTACLE PROBLEMS

We now show how to extend our estimates to solutions of obstacle problems. In particular, our results include some of those in [7] for the capillary problem (specifically Theorem 0.1 there except for the assertion of continuity of the derivatives). To state this problem, we fix a function $\psi \in C^{0,1}$ and set

$$\mathbb{K} = \{v \in C^{0,1}(\bar{\Omega}) : v \geq \psi \text{ in } \Omega\}.$$

Thus we consider $u \in C^{0,1}(\bar{\Omega}) \cap W^{2,2}$ such that

$$\int_{\Omega} A(x, u, Du) \cdot D(u - v) - B(u - v) dx \geq \int_{\partial\Omega} \varphi(u - v) ds$$

for all $v \in \mathbb{K}$. Rather than investigate this variational inequality directly, we study the penalized problem

$$\operatorname{div} A + B - \mu \theta_{\delta}(u - \psi) = 0 \quad \text{in } \Omega, \quad A \cdot \gamma + \varphi = 0 \quad \text{on } \partial\Omega$$

for positive parameters μ and δ , and θ_{δ} a C^1 increasing function with $\theta_{\delta}(t) = 0$ for $t \geq 0$ and $\theta_{\delta}(t) = -1$ for $t \leq -\delta$.

Assuming the existence of solutions to these penalized problems, we shall bound the maxima of $|u|$ and $|Du|$ independently of μ and δ . Moreover we shall show that, for any $\delta \in (0, 1)$ and μ sufficiently large (depending on δ),

$$(6.1) \quad u - \psi \geq -C\delta^{1/2}.$$

(These results are the same as [7, Theorem 1.2].) These estimates imply the regularity of the solution of the variational inequality by standard arguments.

The estimate on u is simple. We have

$$z[B(x, z, p) - \mu\theta_\delta(z - \psi)] \leq zB(x, z, p)$$

for $|z| \geq |\psi|_0$. If $z \geq |\psi|_0$, this is so because $\theta_\delta(z - \psi) = 0$ while for $z \leq 0$, we have $\theta_\delta \leq 0$. Hence the results of §2 gave a bound on $|u|$ independent of μ and δ .

For the gradient bound, we include the term $-\mu\theta_\delta(z - \psi)\delta_k^i$ in D_k^i and observe that we only have to check (3.8c) and (3.11a). Since

$$\begin{aligned} & \left\{ D_i u D_z (-\mu\theta_\delta(z - \psi))\delta_k^i + \frac{\partial}{\partial x^i} (-\mu\theta_\delta(z - \psi)) \right\} \{A^k + \overline{\varphi}\gamma^k\} \\ &= -\mu\theta'_\delta(u - \psi)(D_k u - D_k \psi)(A^k + \overline{\varphi}\gamma^k) \leq 0 \end{aligned}$$

if $v_1 \geq \tau_0$ and $|Du| \geq C(\beta_2, \beta_3)|D\psi|$ by virtue of (3.6), it follows that (3.8c) holds after adding a suitable multiple of $|D\psi|_0$ to τ_0 . The extended version of (3.11a) is obvious because $\theta_\delta \leq 0$.

In some cases, inequality (6.1) can be proved directly from structure conditions analogous to those in §2. (Such is the case, for example, in [7].) Here we take advantage of the gradient bounds just proved. First of all, let us set $w = \psi - u$, $\tilde{\varphi}(x) = \varphi(x, u)$, $\overline{A}(x, z, p) = -A(x, \psi - z, D\psi - p)$, and $\overline{B}(x, z, p) = -B(x, u, Du) + \mu\theta_\delta(-w)$. By suitable extension of \overline{A} , we can find nonnegative constants $\tilde{a}_1, \tilde{b}_1, \tilde{c}_0$ such that

$$p \cdot \overline{A}(x, z, p) \geq |p|^2 - \tilde{a}_1, \quad w\overline{B} \leq \tilde{b}_1, \quad w\varphi \leq \tilde{c}_0$$

wherever $w > 2\delta$. (Of course $\tilde{a}_1, \tilde{b}_1, \tilde{c}_0$ will depend on all the structure used to obtain our gradient bound.) We now apply Lemma 2.1 with $m = 2$, $M = 2\delta$, $q = 2n$, $a_1 = \tilde{a}_1\delta^{-2}$, $b_1 = \tilde{b}_1\delta^{-2}$, $c_0 = \tilde{c}_0\delta^{-1}$, and $b_0 = 0$ to infer that

$$(6.2) \quad \sup_{\Omega} w \leq c(\tilde{a}_1, \tilde{b}_1, \tilde{c}_0, n, q, \Omega) \left[\left(\delta^{-n} \int_{\{w > 2\delta\}} w^{2n} dx \right)^{1/2n} + \delta \right].$$

In fact, for $\mu \geq \tilde{b}_1/\delta$ we have

$$w\overline{B} \leq \frac{1}{2}\mu w,$$

so Lemma 2.2 with $m = 2$, $m_0 = 1$, $M_0 = 2\delta$, $q = 2n$, $a_1 = \tilde{a}_1/\delta$, $b_0 = 0$, $b_1 = \mu/2$, and $c_0 = \tilde{c}_0$ gives

$$(6.3) \quad \int_{\{w > 2\delta\}} w^{2n} dx \leq c(n, \Omega)\delta^{2n}$$

provided μ is sufficiently large (depending also on δ). The combination of (6.2) and (6.3) with the inequality $0 < \delta < 1$ gives (6.1).

When Γ is empty and $\partial\Omega \in C^{2,\alpha}$ for some $\alpha \in (0, 1)$, solutions of the penalized problem exist by virtue of [21, §7] under the additional hypotheses listed there. Standard approximation arguments (see §7) allow us to relax the regularity hypotheses on $\partial\Omega$ and the functions A , B , and φ to those needed for our estimates. In particular, we can allow the singular set Γ to be nonempty.

As in [7, 6, 23], etc., adding the hypotheses

$$\begin{aligned} A(x, \psi, D\psi) \cdot \gamma + \varphi(x, \psi) &\leq 0 \quad \text{on } \partial\Omega, \\ \varphi &\in C^{1,\eta}, B, A_z, A_x \quad \text{in } C^\eta, a^{ij} \in C^0, \partial\Omega \in C^{2,\eta} \end{aligned}$$

for some $\eta \in (0, 1]$, gives second derivative bounds up to $\partial\Omega$.

7. SOME REMARKS ON EXISTENCE

In deriving our gradient bound, we had to assume that the solution was Lipschitz, that is, the gradient was bounded. In the case of a smooth boundary, this assumption is not serious because strong existence results are known; however, for a nonsmooth domain, this assumption is important.

In a convex domain (in two dimensions) near a corner, solutions of conormal derivative problems generally have bounded (and even continuous) gradients while their gradients are generally unbounded near a nonconvex corner. For example, let Ω be the set, in polar coordinates, given by $0 < r < 1$, $0 < \theta < \frac{3\pi}{2}$ and write

$$\Sigma_1 = \left\{ r = 1, 0 < \theta < \frac{3\pi}{2} \right\}, \quad \Sigma_2 = \left\{ 0 < r < 1, \theta = 0, \theta = \frac{3\pi}{2} \right\}.$$

Then $u = r^{2/3} \cos \frac{2}{3}\theta$ is the unique solution up to an additive constant of

$$\Delta u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \gamma} = -\frac{2}{3} \cos \frac{2}{3}\theta \quad \text{on } \Sigma_1, \quad \frac{\partial u}{\partial \gamma} = 0 \quad \text{on } \Sigma_2.$$

At the nonconvex corner $\{r = 0\}$, the gradient is unbounded, while at the convex corners $\{r = 1, \theta = 0 \text{ or } \frac{3\pi}{2}\}$, u has Hölder continuous gradient; in fact, u is C^∞ everywhere except at the origin. The estimates of §3 do not distinguish between the convex and nonconvex corners, but they can only be applied when the solution is known to be Lipschitz.

Two approaches are possible in proving that a solution has bounded gradient. First, the approach used by the author in, e.g., [21] can be applied provided the only singularities of $\partial\Omega$ are edges and the interior angle between the hypersurfaces meeting at each edge is less than π . The only change in the proof is that the linear theory of [22], which is valid in such domains, replaces the classical Schauder theory. Hence, with this additional assumption on the domain, the hypotheses of §§2 and 3 and the conditions

$$(7.1a) \quad A \in C^{1+\alpha}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n), \quad B \in C^\alpha(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n),$$

$$(7.1b) \quad A_p \in C^{1+\delta}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n),$$

$$(7.1c) \quad B_z \text{ and } B_p \text{ are in } C^\delta(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n),$$

$$(7.1d) \quad \varphi \in C^{1+\alpha}(\partial\Omega \times \mathbb{R}),$$

$$(7.2) \quad (a^{ij}) > 0 \quad \text{on } \Omega \times \mathbb{R} \times \mathbb{R}^n,$$

$$(7.3a) \quad A_z = 0, B_z < 0 \quad \text{on } \Omega \times \mathbb{R} \times \mathbb{R}^n,$$

$$(7.3b) \quad \varphi_z \leq 0 \quad \text{on } \partial\Omega \times \mathbb{R}$$

for some positive α, δ imply that (0.1) has a unique $C^{1+\varepsilon}(\overline{\Omega})$ solution for suitable $\varepsilon > 0$. The local results in §3 are useful in this case primarily for showing that the bound on the gradient depends only on local data. As pointed out in [21], conditions (7.1), (7.2), and (7.3) can be modified somewhat. Thus, if the only singularities of $\partial\Omega$ are edges and if conditions (2.1)', (3.1)–(3.12), (7.1), (7.2), and (7.3) hold, then (0.1) has a unique solution, which has bounded gradient. (Of course if $m = 1$ in (2.1)', we should replace (2.1c)' by (2.1c)'').

An alternative approach is needed to consider domains with other types of singularities. The approximation scheme sketched by Korevaar in [9, §4] is appropriate for this situation.

From the appendix there is a function $\rho \in C^{0,1}(\overline{\Omega}) \cap C^\infty(\Omega)$ which is equivalent to d . Since Ω satisfies an exterior sphere condition and an interior cone condition, there are positive constants ε_0 and ρ_0 such that $|D\rho| > \rho_0$ for $0 < \rho < \varepsilon_0$ and, if $\Omega_\varepsilon = \{x \in \Omega : \rho(x) > \varepsilon\}$ with inner normal γ_ε , we have $\partial\Omega_\varepsilon \in C^\infty$, $\overline{\gamma} \cdot \gamma_\varepsilon \geq C(\Omega)\gamma_0$, and

$$D_i \gamma_{\varepsilon,j} \zeta^i \zeta^j \leq \frac{20}{\rho_0 R} |\zeta|^2$$

for $\zeta \cdot \gamma_\varepsilon = 0$ and $0 < \varepsilon < \varepsilon_0$. Hence if conditions (2.1)', (3.1)–(3.12), (7.1a, b, c), (7.2), (7.3) and

$$(7.1d)' \quad \overline{\varphi} \in C^{1+\alpha}(\partial\Omega \times \mathbb{R})$$

hold for (0.1), the boundary value problem

$$\begin{aligned} \operatorname{div} A(x, u_\varepsilon, Du_\varepsilon) + B(x, u_\varepsilon, Du_\varepsilon) &= 0 \quad \text{in } \Omega_\varepsilon, \\ A(x, u_\varepsilon, Du_\varepsilon) \cdot \gamma_\varepsilon + \overline{\gamma} \cdot \gamma_\varepsilon \overline{\varphi} &= 0 \quad \text{on } \partial\Omega_\varepsilon \end{aligned}$$

has a unique solution $u_\varepsilon \in C^{2+\alpha}(\overline{\Omega}_\varepsilon)$. From §3, $|Du_\varepsilon|$ is bounded uniformly as $\varepsilon \rightarrow 0$, so u_ε converges uniformly to the unique solution of (0.1). Standard regularity shows that $u \in C^{2+\alpha}(\Omega)$, it follows from [19] that $u \in C^{1+\theta}(\overline{\Omega} \setminus \Gamma)$ for $\theta \in (0, 1)$, and [20] implies that Du is Hölder continuous near interior points of edges of $\partial\Omega$ as well.

APPENDIX. REMARKS ON REGULARIZED DISTANCE

In this appendix, we construct a regularized distance for Ω , i.e., a function $\rho \in C^{0,1}(\mathbb{R}^n) \cap C^2(\mathbb{R}^n \setminus \partial\Omega)$ which is equivalent to the signed distance function d defined by

$$d(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x \notin \Omega. \end{cases}$$

To prove our results, we recall the construction of regularized distance from [14, Lemma 1.1] with a slight modification to take account of a more detailed

structure. Let φ be a nonnegative $C^2(\mathbb{R}^n)$ function supported in the unit ball with $\int \varphi(z) dz = 1$. If g is a Lipschitz function such that g/d and d/g are uniformly bounded and positive in $\mathbb{R}^n \setminus \partial\Omega$, and L and M are positive constants such that $|g(x) - g(y)| \leq M|x - y|$ for all x and y in \mathbb{R}^n and $L > M$, we define

$$(A.1) \quad G(x, \tau) = \int g(x - \tau z/L) \varphi(z) dz$$

for $(x, \tau) \in \mathbb{R}^{n+1}$. A regularized distance ρ is then given by the equation $\rho(x) = G(x, \rho(x))$. To see that this equation defines ρ uniquely, we note from formulas (1.3) and (1.4) in [14] that the partial derivatives G_x and G_τ satisfy the inequalities

$$(A.2) \quad |G_x(x, \tau)| \leq M, \quad |G_\tau(x, \tau)| \leq M/L < 1.$$

We also note that

$$(A.3) \quad D\rho(x) = \frac{G_x(x, \rho(x))}{1 - G_\tau(x, \rho(x))}.$$

We now present two definitions. If $x_0 \in \partial\Omega$ and if there are positive constants δ, A , an orthonormal coordinate system $Y = (y', y'') = (y^1, \dots, y^n)$, and a function $f: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ with

$$|f(y'_1) - f(y'_2)| \leq A|y'_1 - y'_2| \quad \text{for } |y'_1|, |y'_2| < 4\delta$$

such that

$$\{y \in \Omega : |y| < 4\delta\} = \{y \in \mathbb{R}^n : y'' > f(y'), |y| < 4\delta\},$$

we call f a local representation of $\partial\Omega$ near x_0 . If $\partial\Omega$ has a local representation near each point x_0 of $\partial\Omega$ with constants δ and A independent of x_0 , we say Ω is Lipschitz. (More commonly, such a domain is called uniformly Lipschitz.) Note that a domain is Lipschitz if and only if it satisfies uniform interior and exterior cone conditions.

Lemma A.1. *Any Lipschitz domain Ω has a proper regularized distance ρ . Moreover, if $0 < |d(x)| < \delta$, then $\frac{3}{2}|D\rho(x)| \geq (1 + A^2)^{-1/2}$.*

Proof. Let $x \in \Omega$ with $d(x) < \delta$ and choose $x_0 \in \partial\Omega$ so that $|x - x_0| = d(x)$. Working in the Y coordinate system corresponding to x_0 , we first show that if $|y_1|, |y_2| < 2\delta$ with $y'_1 = y'_2$ and $y''_1 > y''_2 \geq f(y'_1)$, then

$$(A.4) \quad d(y_1) - d(y_2) \geq \frac{y''_1 - y''_2}{(1 + A^2)^{1/2}}.$$

To prove (A.4), let $y_0 \in \partial\Omega$ such that $d(y_1) = |y_1 - y_0|$, note that $|y'_0| < 4\delta$, and let K be the exterior cone to Ω with vertex y_0 , vertex angle $2 \operatorname{arccot} A$, and axis parallel to the y'' -axis. Then

$$(A.5) \quad d(y_1) = \operatorname{dist}(y_1, K), \quad d(y_2) \leq \operatorname{dist}(y_2, K).$$

Now write \mathcal{L} for the line through y_2 and parallel to the line segment $y_0 y_1$, z_0 for the point of intersection of \mathcal{L} with the axis of K and z_1 for the point of intersection of \mathcal{L} with K , which is closer to y_2 . It follows that the cotangent of the angle $z_0 y_0 z_1$ is A , so $|z_0 - z_1| \geq (1 + A^2)^{-1/2} |y_0 - z_0|$. Since

$|y_0 - z_0| = y_1^n - y_2^n$, $\text{dist}(y_2, K) \leq |y_2 - z_1|$, and $\text{dist}(y_1, K) = |y_2 - z_0| = |y_2 - z_1| + |z_1 - z_0|$, we also have

$$(A.6) \quad \text{dist}(y_1, K) - \text{dist}(y_2, K) \geq \frac{y_1^n - y_2^n}{(1 + A^2)^{1/2}}.$$

The combination of (A.5) and (A.6) gives (A.4).

Now take $g = d$ and $M = 1$ in (A.1). If $X' = x'$ and $f(x') < X^n < x^n$, then

$$G(x, \tau) - G(X, \tau) = \int \left[d\left(x - \frac{\tau z}{L}\right) - d\left(X - \frac{\tau z}{L}\right) \right] \varphi(z) dz \geq \frac{x^n - X^n}{(1 + A^2)^{1/2}}$$

provided $0 < \tau < L \min\{d(X), d(x)\}$. Hence if we divide this inequality by $x^n - X^n$ and let X^n approach x^n , we find that the derivative $G_n = \partial G / \partial x^n$ satisfies the estimate $G_n(x, \tau) \geq (1 + A^2)^{-1/2}$ for $0 < \tau < Ld(x)$, and by continuity of G_n , also for $\tau = Ld(x)$. In particular, we can use $\tau = \rho(x)$ to infer that

$$(A.7) \quad G_n(x, \rho(x)) \geq (1 + A^2)^{-1/2}.$$

From (A.2), (A.3), and (A.7), we have $D_n \rho(x) \geq \frac{L}{(L+M)(1+A^2)^{1/2}}$ for $0 < d(x) < \delta$, and a similar argument works if $-\delta < d(x) < 0$. Choosing $L = 2$ gives the estimate on $|D\rho|$. \square

Note that by choosing L sufficiently large, we can make $|D\rho| \geq \alpha(1 + A^2)^{-1/2}$ for any $\alpha \in (1/2, 1)$.

The method of [14, §2] gives an alternative proof of Lemma 1 with different constants by constructing local regularized distances and then piecing them together via a partition of unity.

When the exterior cone condition is strengthened to an exterior sphere condition, the regularized distance just constructed satisfies a one-sided second derivative condition. The proof of this fact uses the following consequence of [14, (1.10)]: Let $x \in \mathbb{R}^n \setminus \partial\Omega$ and $\xi \in \mathbb{R}$, set $\Xi = (\xi, \xi \cdot D\rho(x))$, and use subscripts $i = 1, \dots, n+1$ on G to denote partial derivatives with respect to x^i if $i \leq n$ and with respect to τ if $i = n+1$. Then

$$(A.8) \quad D_{ij} \rho(x) \xi^i \xi^j = \frac{G_{ij}(x, \rho(x)) \Xi^i \Xi^j}{1 - G_{n+1}(x, \rho(x))}.$$

(Here and below we follow the summation convention that repeated indices are to be summed over the appropriate range. For example, in (A.8) the indices i and j are summed from 1 to n on the left-hand side but from 1 to $n+1$ on the right.)

Theorem A.2. *If Ω satisfies a uniform exterior sphere condition with radius R , then Ω has a regularized distance ρ such that*

$$(A.9) \quad D_{ij} \rho(x) \xi^i \xi^j \leq \frac{C(n)}{R} |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and all $x \in \mathbb{R}^n \setminus \partial\Omega$, and

$$(A.10) \quad D_{ij} \rho(x) \xi^i \xi^j \leq \frac{4}{R} |\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and all $x \in \Omega$. If Ω satisfies a uniform interior cone condition, then ρ is proper.

Proof. We first prove that

$$(A.11) \quad d(x+h) + d(x-h) - 2d(x) \leq \frac{|h|^2}{R}$$

for all $x \in \Omega$ and all $h \in \mathbb{R}^n$ with $|h| < d(x)$. To prove this inequality, for each $y \in \partial\Omega$, write z for the center of the exterior sphere to Ω at y with radius R , and define d_y by $d_y(x) = |x - z| - R$. Note that $d \leq d_y$ and $d_y \in C^2(\mathbb{R}^n \setminus \{z\})$ for any $y \in \partial\Omega$. Therefore

$$D_{ij}d_y(x)h^ih^j = \frac{|h|^2}{|x-z|} - \frac{|(x-z) \cdot h|^2}{|x-z|^3} \leq \frac{|h|^2}{|x-z|} \leq \frac{|h|^2}{R}.$$

For $|h| < d_y(x)$, Taylor's Theorem shows that $d_y(x+h) + d_y(x-h) - 2d_y(x) \leq \frac{|h|^2}{R}$. If we choose y so that $d(x) = |x - y|$, then $d(x) = d_y(x)$. Since $d_y \leq d$, (A.11) follows immediately.

From (A.11) and (A.1) with $g = d$ and φ even, so that $\int z\varphi(z)dz = 0$, we see that

$$G(x+h, \tau+k) + G(x-h, \tau-k) - 2G(x, \tau) \leq \frac{|h|^2 + k^2/L^2}{R}$$

for $0 < \tau < Ld(x)/2$ and $|h| + |k|/L \leq d(x)/2$. Dividing by $|h|^2 + k^2/L^2$, letting this denominator tend to zero, and choosing $\tau = \rho(x)$ gives

$$(A.12) \quad G_{ij}(x, \rho(x))\Xi^i\Xi^j \leq \frac{|\Xi'|^2 + (\Xi^{n+1}/L)^2}{R}$$

for any $\Xi = (\Xi', \Xi^{n+1}) \in \mathbb{R}^{n+1}$. It now follows from (A.2), (A.3), and (A.8) that

$$D_{ij}\rho(x)\xi^i\xi^j \leq \frac{L(|\xi|^2 + |\xi \cdot D\rho(x)|^2)/L^2}{(L-M)R} \leq \frac{4|\xi|^2}{R}$$

for $L = 2$, $M = 1$.

When $-R/2 < d(x) < 0$, we have $|x - z| \geq R/2$ and hence the preceding argument shows that $D_{ij}\rho(x)\xi^i\xi^j \leq \frac{8}{R}|\xi|^2$.

Finally, it follows from the proof of [14, Theorem 1.3] that $|D^2\rho| \leq C'(n)/d$, and therefore $d(x) < -R/2$ implies that (A.9) holds with C replaced by C' . Hence (A.9) holds in all cases if $C = \max\{8, C'\}$.

Lemma A.1 implies that ρ is proper in case of an interior cone condition. \square

By choosing L sufficiently large, we can replace 4 in inequality (A.10) by any constant greater than 1.

We now show how to infer (3.12)' from (3.12). Suppose that Ω is a Lipschitz domain satisfying an exterior sphere condition with radius R and let ρ be the regularized distance constructed in Lemma A.1. With δ as in the definition of Lipschitz domain and $\varepsilon \in (0, \delta)$, write γ for the normal to the surface $\Sigma = \{x : \rho(x) = \varepsilon\}$ and let ξ be a vector field tangent to Σ . Since $\gamma = D\rho/|D\rho|$ on Σ , we see from (A.3), (A.7), (A.8), and (A.12) that

$$\begin{aligned} D_i\gamma_j\xi^i\xi^j &= \frac{1}{|D\rho|}D_{ij}\rho\xi^i\xi^j = \frac{1}{|G_x(x, \rho(x))|}G_{ij}(x, \rho(x))\Xi^i\Xi^j \\ &\leq (1+A^2)^{1/2}\frac{|\Xi'|^2}{R} = (1+A^2)^{1/2}\frac{|\xi|^2}{R} \quad \text{on } \Sigma. \end{aligned}$$

In a sufficiently small neighborhood of $x_0 \in \partial\Omega \setminus \Gamma$, we can make A arbitrarily close to zero by choosing δ small enough. Hence for any constant $C > 1$, we have $D_i \gamma_j \zeta^i \zeta^j \leq C \frac{|\zeta|^2}{R}$, and sending C to 1 proves (3.12)'.

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